Lifting-Surface Theory for a Semi-Infinite Wing in Oblique Gust

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An unsteady lifting-surface theory is developed for the calculation of the airload on a semi-infinite-span thin wing in a compressible flow due to interaction with an oblique gust. By using the solutions obtained for a two-dimensional wing, the problem is formulated so that the unknown is taken to be the difference between the airload on the semi-infinite wing and that on a two-dimensional wing under the same gust conditions. Since this airload difference is nonzero only near the wing tip, the control points need be distributed in the tip region only; this significantly simplifies the numerical procedure. Results are presented for a wing with rectangular tip. The implication for noise and unsteady loads due to blade-vortex interaction for helicopter rotors is discussed.

Nomenclature

A_p = multiplying factor for pressure distribution, see Appendix B
a = speed of sound
b_o = wing semichord
b_T = wing semichord at modified tip region
C_L = lift coefficient
C_p = pressure coefficient
ΔC_{p_c} = airload difference near wing tip [see Eq. (12)]
$f_n(x), g_m(y)$ = chordwise and spanwise loading functions
$\mathscr{F}$ = tip-loading function ($= C_L/C_{L\infty}$)
$\bar{h}$ = vertical spacing between blade and vortex, normalized by b_o
$I_n(x), K_n(x)$ = nth-order modified Bessel functions of first and second kind
$K(\bar{x}_o, \bar{y}_o)$ = kernel function
$\bar{K}(\bar{x}_o, \bar{y}_o)$ = modified form of kernel function
k = gust wave number
k_o = normalized gust wave number ($= kb_o$)
k_x, k_y = chordwise and spanwise components of gust wave number
$\bar{k}_y$ = $k_y l_y$
l_y = characteristic length in spanwise direction
$L_1(x)$ = modified Struve function
$L_m(x)$ = mth Laguerre polynomial
M = freestream Mach number
s = reduced frequency ($= \omega b_o/U$)
$T_{o\infty}$ = aerodynamic transfer function for two-dimensional wing in incompressible flow
t = time
$\bar{t}$ = normalized time ($= tU/b_o$)
U = freestream velocity
w = gust upwash velocity
x, y, z = Cartesian coordinates attached to wing
$\bar{x}, \bar{y}$ = x/b_o, y/l_y
x_c = midchord location for modified tip region
$\bar{x}_o, \bar{y}_o$ = $\bar{x}-\bar{x}_1$, $\bar{y}-\bar{y}_1$
x_L, x_T = locations of leading edge and trailing edge
y_T = range of arbitrary tip planform
$\bar{y}_T$ = y_T/l_y
α = gust-induced angle of attack
β = $(1-M^2)^{1/2}$
Γ = circulation of vortex
θ_c = angular chordwise variable
Λ = wing-gust interaction angle
μ = ratio of characteristic lengths in spanwise and chordwise directions ($= l_y/b_o$)
ϕ = perturbation velocity potential
ϕ_T = phase of tip loading function
ω = radian frequency

Subscripts
$(\ )_o$ = property for incompressible flow
$(\ )_\infty$ = property for two-dimensional wing

Superscripts
$(\bar{\ })$ = $f = \bar{f}\exp(-i\omega t)$ [see Eqs. (6) and (7)]
$(\hat{\ })$ = $f = \hat{f}\exp[i(k_x x + k_y y - \omega t)]$ [see Eq. (1)]

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Introduction

THE evaluation of the unsteady airload induced on a thin wing interacting with turbulent gusts has always been an important problem in aerodynamics because of its broad applications. Practical examples include the calculation of dynamic airloads on fixed and rotary wings and the prediction of aerodynamic noise from unsteady forces. With the assumption that the magnitude of the disturbance velocity is small in comparison with the freestream velocity, the problem can be linearized. The problem considered here is that of a flat, thin wing interacting with an oblique sinusoidal gust in a compressible flow. The gust is taken to be convected with the freestream. By linearity, it is then possible to separate the response to any three-dimensional gust into the responses to a spectrum of oblique sinusoidal gusts through Fourier-transform technique; both the gust and the induced airload are then harmonic in time.

One of the classical problems in this category is the wing-gust interaction problem solved analytically by Sears[1] for incompressible flow: a two-dimensional wing that interacts with a sinusoidal gust convected with the freestream having its wave front parallel to the leading edge of the wing. For the three-dimensional interaction of a thin wing and an oblique gust, the situation is more complicated. If the aspect ratio of the wing is very large, it has been customary to use lifting-line theory.[2] It is assumed that the flow over the wing is locally two dimensional, and the influence of the rest of the wing and the wake is represented only by a uniform downwash along the wing chord. Two-dimensional, unsteady airfoil theory is then used to obtain the airloads. However, if the variation of the gust upwash along the span is large, the effective aspect ratio—the ratio of the gust semiwavelength along the span to the wing chord—is small. Therefore, as pointed out by Johnson,[3] the lifting-line theory is not valid, and it is necessary to use the more accurate lifting-surface theory.

To calculate unsteady loadings on the midportion of a large-aspect-ratio wing, the effect of the wing tip is relatively unimpor-

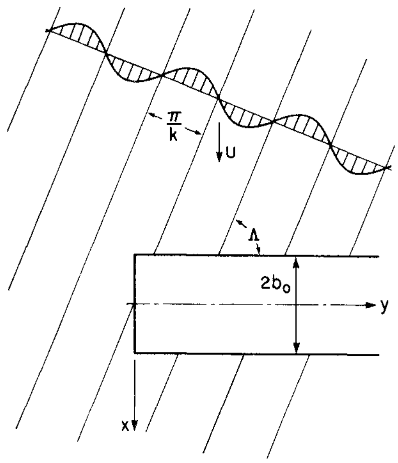

Fig. 1 Semi-infinite-span thin wing interacting with a sinusoidal gust.

tant. Therefore, the theoretical model wing can be taken to be two-dimensional with infinite span. The airload on the wing would then repeat itself cyclically and can be considered to be steady by choosing a coordinate system which convects along the span with the gust trace-wave speed. Filotas[4] considered this three-dimensional problem for incompressible flow and derived approximate closed-form solutions for both the pressure distribution and the section lift. Graham[5] treated the same problem by using a numerical-series method; and later developed a similarity rule to extend his numerical results to compressible flow.[6] Johnson[3] approached the problem directly from the general compressible-flow equation. The sweepback of the wing is also included in his analysis. By using the GASP approximation derived by Amiet and Sears,[7] Chu and Widnall[8] have extended Filotas' incompressible solution to compressible flow.

However, if one is concerned with airload induced near the wing tip, such as that due to blade-vortex interaction at the tip of a helicopter rotor blade, the solution obtained by using a two-dimensional wing as the theoretical model cannot provide satisfactory results for the airload calculations. To describe the flow near the wing tip, a fully unsteady, three-dimensional lifting-surface theory has to be developed. One of the standard procedures for formulating the lifting-surface problem, as suggested by Küssner,[9] is to convert the boundary value problem into an integral equation that relates a prescribed upwash distribution to an unknown lift distribution. The advantage of this formulation is that the lift distribution is zero in the wake; therefore, the integration need be carried out only over the wing surface. Practical methods for solving the integral equation, using assumed loading functions to convert the integral equation into a set of linear algebraic equations, were developed by Watkins, Woolston, and Cunningham.[10] Recent progress in numerical lifting-surface theories has been summarized and discussed by Landahl and Stark.[11]

The difficulty in formulating the lifting-surface problem for a large-aspect-ratio wing is that the wing span is so large that it is practically impossible to distribute control points over the entire wing and solve the problem in the conventional way. However, since the span of the wing is large, we may assume that the airload induced near one tip is hardly influenced by the other tip. Therefore, we can use a semi-infinite-span wing as the theoretical model. If the unknown is taken to be the difference between the load on the semi-infinite wing and that on an infinite wing, the numerical problem can be simplified significantly. This paper provides a detailed discussion of the numerical method developed for this problem.

Additionally, we discuss the numerical results of our calculations for various values of the gust wave number and the interaction angle. Even though the application of the present results to calculate rotary wing airloads is somewhat limited because of the neglect of the velocity variation along the span and the curvature of the gust induced by skewed wakes, it should well serve the purpose of calculating airloads for interactions which are of importance for only a small portion of the wing and a short period of time. The blade-vortex interaction at the tip of an advancing helicopter rotor blade provides a good example for this case. A calculation of unsteady airloads due to such blade-vortex interaction is included to show the application of the results.

Formulation of the Theoretical Problem

The theoretical model we chose is a rigid, flat, thin wing with semi-infinite span and a square tip at one end. The chord of the wing is taken to be $2b_o$. The wing is flying with speed U and interacting with an oblique sinusoidal gust that is stationary in the air. The gust is described by its wave number k and the interaction angle Λ (Fig. 1). The upwash velocity of the sinusoidal gust can be written as

$$w(x, y, t) = \hat{w}\exp[i(k_x x + k_y y - \omega t)] \tag{1}$$

with $k_x = k\cos\Lambda$ and $k_y = k\sin\Lambda$, which are the wave number components in the x and y directions, and $\omega = kU\cos\Lambda$, the radian frequency.

For small perturbations, the velocity field due to the thin wing is irrotational and may be linearly combined with the gust velocity field, even though the gust flow is vortical. In terms of the perturbation velocity potential function $\phi(x, y, z, t)$, the total velocity components in the flowfield are $U+\phi_x$, ϕ_y, and $w+\phi_z$. The linearized governing equation for the flow is

$$\nabla^2\phi - \frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi = 0 \tag{2}$$

and the perturbation pressure coefficient is given by

$$C_p = -\frac{2}{U}\left(\frac{\partial\phi}{\partial x} + \frac{1}{U}\frac{\partial\phi}{\partial t}\right) \tag{3}$$

The boundary conditions can be separated into three parts:
1) The flow is tangent to the wing surface

$$\phi_z|_{z=0\pm} = -\hat{w}\exp[i(k_x x + k_y y - \omega t)] \quad \text{for} \quad |x| \leqq b_o,\ y \geqq 0 \tag{4}$$

2) The pressure must be continuous both at the trailing edge (Kutta condition) and in the wake

$$\Delta C_p(z = 0\pm) = 0 \quad \text{for} \quad x \geqq b_o \tag{5}$$

3) The amplitude of the disturbance approaches zero at infinity, except in the wake.

Since the gust upwash is harmonic in time, so should the velocity potential and the induced airload, i.e.,

$$\phi(x, y, z, t) = \bar{\phi}(x, y, z)\exp(-i\omega t) \tag{6}$$

$$\Delta C_p(x, y, z = 0\pm, t) = \Delta\bar{C}_p(x, y, z = 0\pm)\exp(-i\omega t) \tag{7}$$

By using the procedures suggested by Küssner,[9] this boundary value problem can be converted to a singular integral equation which relates the unknown airload distribution and the downwash on the wing surface; namely

$$-\frac{\hat{w}}{U}\exp[i(s\bar{x} + \bar{k}_y\bar{y})] = \frac{1}{8\pi}\not\int_0^\infty \int_{-1}^{1} \Delta\bar{C}_p(\bar{x}_1, \bar{y}_1) \times K(\bar{x}_o, \bar{y}_o)\, d\bar{x}_1\, d(\mu\bar{y}_1) \tag{8}$$

where μ is the ratio of the spanwise characteristic length l_y to the chordwise characteristic length b_o. From the geometry of the problem, it would be expected that l_y is a function of both the wing chord and the gust wavelength. The numerical value of l_y or μ is not taken at this stage. This choice will be taken and discussed in later analyses.

In Eq. (8), $K(\bar{x}_o, \bar{y}_o)$ is the normalized kernel function, which represents the downwash induced at the point $(\bar{x}, \bar{y}, 0)$ by a single pulsating dipole located at $(\bar{x}_1, \bar{y}_1, 0)$. Its expression can be found in Ref. 9. Watkins, Runyan, and Woolston[12] have derived a working form of the kernel function, which is given as

$$K(\bar{x}_o, \bar{y}_o) = \bar{K}(\bar{x}_o, \bar{y}_o)/\mu^2\bar{y}_o^2 \tag{9}$$

with $\bar{K}(\bar{x}_o, \bar{y}_o)$ as the modified form of the kernel function whose expression is given in Appendix A. $\bar{K}(\bar{x}_o, \bar{y}_o)$ has no infinite singularities, but it does have a finite jump at $\bar{x}_o = \bar{y}_o = 0$, i.e., at the position of the dipole

$$\lim_{\bar{y}_o \to 0} \bar{K}(\bar{x}_o, \bar{y}_o) = \begin{cases} 0 & \bar{x}_o < 0 \\ -2\exp(is\bar{x}_o) & \bar{x}_o > 0 \end{cases} \tag{10}$$

Therefore, the kernel function $K(\bar{x}_o, \bar{y}_o)$ has a second-order singularity when $\bar{y}_o = 0$ and $\bar{x}_o \geqq 0$, i.e., it is singular at the position of the dipole, and in the wake extending from the dipole downstream to infinity. Hence, in evaluating surface integrals over the wing surface, the finite part in the sense of Mangler[13] is taken for the integral over the singularity. Watkins, Woolston, and Cunningham[10] developed an approximate form for the modified kernel function, which is used in the numerical programing; this expression is also given in Appendix A.

The usual procedure for solving the integral equation is to assume the loading to be a series of preselected chordwise and spanwise loading functions with unknown coefficients, and then to determine them by satisfying the normal velocity condition at a set of collocated control points on the wing surface.[10] For our problem here, because of the special geometry, modifications have to be made to simplify the mathematical procedure.

The particular feature of this problem that differs from conventional lifting-surface theories is that, since the wing has a semi-infinite span, it is impossible to distribute the control points evenly over the entire wing surface. The problem is formulated rather in terms of the difference between the loading on the semi-infinite wing and that on a two-dimensional wing under the same gust conditions. Since the three-dimensional effect of the wing tip is reduced as the point of interest moves inboard, this load difference is nonzero only near the tip. Therefore, we can restrict the distribution of control points only at the tip region. The integral equation can then be modified to the form

$$-\frac{\hat{w}}{U}\exp[i(s\bar{x}+\bar{k}_y\bar{y})] = \frac{\mu}{8\pi} \int_0^\infty \int_{-1}^1 (\Delta\bar{C}_{p_\infty} + \Delta\bar{C}_{p_c}) \times K(\bar{x}_o, \bar{y}_o)\, d\bar{x}_1\, d\bar{y}_1 \tag{11}$$

where $\Delta\bar{C}_{p_\infty}$ represents the airload distribution of the two-dimensional wing; its expression from Ref. 8 is given in Appendix B; and

$$\Delta\bar{C}_{p_c} \equiv \Delta\bar{C}_p - \Delta\bar{C}_{p_\infty} \tag{12}$$

which represents the airload difference near the tip. From Eq. (12), it is apparent that $\Delta\bar{C}_{p_c}$ should have the asymptotic behavior

$$\lim_{\bar{y} \to \infty} \Delta\bar{C}_{p_c}(\bar{x}, \bar{y}) = 0 \tag{13}$$

and $\Delta\bar{C}_{p_\infty}$ should satisfy the integral equation for the two-dimensional wing; namely,

$$-\frac{\hat{w}}{U}\exp[i(s\bar{x}+\bar{k}_y\bar{y})] = \frac{\mu}{8\pi} \int_{-\infty}^\infty \int_{-1}^1 \Delta\bar{C}_{p_\infty}(\bar{x}_1, \bar{y}_1) \times K(\bar{x}_o, \bar{y}_o)\, d\bar{x}_1\, d\bar{y}_1 \tag{14}$$

By eliminating common terms in Eqs. (11) and (14), the equation can be further simplified to the form

$$\int_{-\infty}^0 \int_{-1}^1 \Delta\bar{C}_{p_\infty}(\bar{x}_1, \bar{y}_1) K(\bar{x}_o, \bar{y}_o)\, d\bar{x}_1\, d\bar{y}_1 = \int_0^\infty \int_{-1}^1 \Delta\bar{C}_{p_c}(\bar{x}_1, \bar{y}_1) K(\bar{x}_o, \bar{y}_o)\, d\bar{x}_1\, d\bar{y}_1 \tag{15}$$

Equation (15) is the operational form of the integral equation. Physically, this can be interpreted to mean that when a half of a two-dimensional wing is moved away, the effect on the induced upwash should be compensated by an increase of the loading on the rest of the wing; and this load increase is distributed near the tip. The left-hand side of the equation, which shows the upwash induced by cutting away half of the wing, is a direct integration. This is the "direct problem." The value obtained from this direct integration and the right-hand side of Eq. (15) forms the integral equation for the load increase, $\Delta\bar{C}_{p_c}$. This is the "indirect problem."

Direct Problem

By substituting the two-dimensional airload distribution (Appendix B) in Eq. (15), the integral of the direct problem is in the form

$$8\pi\left(\frac{-\Delta\bar{w}}{U}\right) = A_p \int_{-\infty}^0 \int_{-1}^1 \left[\frac{1-\bar{x}_1}{1+\bar{x}_1}\right]^{1/2} \exp(-K'\bar{x}_1)\bar{K}(\bar{x}_o, \bar{y}_o)\, d\bar{x}_1 \cdot \left[\frac{\exp(i\bar{k}_y\bar{y}_1)}{\mu\bar{y}_o^2}\right] d\bar{y}_1 \tag{16}$$

Since the control points are collocated only on the wing surface with $\bar{y} > 0$, $\bar{y}_o$ is always larger than zero. Therefore, the only singularities in the integrand are those at the leading edge. The multi-integral can then be evaluated immediately by using numerical quadrature formulas.

For the chordwise integration, the Jacobi-Gauss quadrature formula with weighting function $[(1-\bar{x})/(1+\bar{x})]^{1/2}$ is used.[14,15] By using this formula, the singularity at the leading edge has automatically been taken care of and no special treatment is necessary. Since the kernel function varies abruptly near $\bar{x}_o = 0$ while $|\bar{y}_o|$ is small,[10] more points should be taken in using the quadrature formula for such cases. We chose the number of the chordwise integration points to be 10 for $s\mu|\bar{y}_o| > 0.3$, and 20 otherwise, as suggested in Ref. 10.

For spanwise integration, since the integration interval is from $-\infty$ to 0, we may first make a change of variable so that the interval is from 0 to ∞, then use the Laguerre-Gauss quadrature formula.[14,15] The difficulty of this numerical integration is that the integrand oscillates with $\exp(i\bar{k}_y\bar{y}_1)$ and decays very slowly as $\bar{y}_1$ approaches infinity, because $\Delta\bar{C}_{p_\infty}$ has a constant amplitude. The most straightforward way to treat this problem is to increase the number of integration points to such a degree that both the density and the range of the integration-points distribution are large enough that both difficulties can be eliminated. However, the computing time needed for this approach would be increased to such an extent that, from an engineering point of view, it is practically infeasible.

Here, we eliminate the difficulties by expanding the kernel function in its asymptotic form so that the leading terms can be integrated analytically; only the remainder, which will decay much more rapidly, needs to be integrated numerically. The kernel function given in Appendix A is expanded term by term. Depending on the decay rate of the remainder for each expansion, as well as the wavelength of the spanwise oscillation, the number of the integration points is chosen to be 24, 32, or 40.

Indirect Problem

In the indirect problem, the airload increase near the wing tip is evaluated by solving the following Fredholm integral equation:

$$8\pi\left(\frac{-\Delta\bar{w}}{U}\right) = \int_0^\infty \int_{-1}^1 \Delta\bar{C}_{p_c}(\bar{x}_1, \bar{y}_1)\bar{K}(\bar{x}_o, \bar{y}_o)\, d\bar{x}_1 \frac{d\bar{y}_1}{\mu\bar{y}_o^2} \tag{17}$$

with the left-hand side of the equation having been obtained from the direct problem. The unknown airload distribution on the wing is replaced by a series of chordwise and spanwise loading functions, with their weighting factor as undetermined coefficients

$$\Delta\bar{C}_p(\bar{x}, \bar{y}) = \sum_n \sum_m c_{nm}\, g_m(\bar{y}) f_n(\bar{x}) \tag{18}$$

The choice of the loading functions to represent the unknown airload distribution should be made with discretion because the more compatible these functions are with the actual loading, the fewer will be required. As a guide in selecting the loading functions, the character of the lift distribution, at least in the neighborhood of the wing edges, can be surmised from the knowledge of a few exact solutions to the lifting-surface problem.[10] For subsonic flow, the pressure distribution on the wing should approach zero along the trailing edge (Kutta condition) and side edge as $\lim_{\varepsilon\to 0}\varepsilon^{1/2}$, where ε is the distance to the wing edge.[10]

At the leading edge, the pressure distribution should behave as $\lim_{\varepsilon \to 0} \varepsilon^{-1/2}$.[10] We also know that for the semi-infinite wing, the effect of the wing tip is reduced as the point of interest moves inboard; thus, the load distribution should approach that for a two-dimensional wing. Based on these conditions, we may choose the chordwise and the spanwise loading functions.

Chordwise Loading Functions

The zeroth chordwise loading function we chose is the chordwise pressure distribution for a two-dimensional wing under the same gust conditions (Appendix B)

$$f_0(\bar{x}) = \left[\frac{1-\bar{x}}{1+\bar{x}}\right]^{1/2} \exp(-K'\bar{x}) \tag{19}$$

Introducing the angular chordwise variable θ_c, defined as

$$\theta_c \equiv \cos^{-1}(-\bar{x}) \tag{20}$$

the zeroth chordwise loading function can then be written as

$$f_0(\theta_c) = \cot(\theta_c/2)\exp(K'\cos\theta_c) \tag{21}$$

The higher-order chordwise loading functions are chosen to be

$$f_n(\theta_c) = \sin(n\theta_c) \tag{22}$$

for n equal to and larger than 1.

Since the zeroth chordwise loading function is of the form of the asymptotic limit of the chordwise pressure distribution at locations far inboard, we would expect the higher-order loading functions would be of importance only near the wing tip.

These chordwise loading functions are basically the same as suggested by Watkins, Woolston, and Cunningham,[10] except for the modified form of the zeroth chordwise loading function due to our knowledge of the asymptotic behavior of the load distribution under oblique gusts.

Spanwise Loading Functions

Since the model wing considered here has a semi-infinite span, the choice of the spanwise loading functions will be different from the earlier works on lifting-surface problems for finite wings. In order to satisfy the side-edge condition and the asymptotic behavior at far inboard, these loading functions should behave as

$$\lim_{\bar{y}\to 0} g_m(\bar{y}) \sim \lim_{\bar{y}\to 0} \bar{y}^{1/2} = 0 \qquad \text{for} \qquad m \geqq -1 \tag{23}$$

and

$$\lim_{y\to\infty} g_m(\bar{y}) = \begin{cases} \exp(i\bar{k}_y y) & \text{for} \quad m=-1 \\ 0 & \text{for} \quad m \geq 0 \end{cases} \tag{24}$$

The loading function for $m = -1$ is the basic spanwise lift distribution. At far inboard locations, it should be able to converge to the sinusoidal spanwise load distribution for a two-dimensional wing. We chose it to be

$$g_{-1}(\bar{y}) = [(2\pi\bar{y})^{1/2}\exp(-\bar{y})I_0(\bar{y})]\exp(i\bar{k}_y\bar{y}) \tag{25}$$

where $I_o(\bar{y})$ is the zeroth modified Bessel function of the first kind. The choice of the basic spanwise lift distribution is arbitrary, provided the side-edge condition and the asymptotic behavior condition are satisfied. It can be shown that Eqs. (23) and (24) are satisfied by the expression given in Eq. (25). The coefficient of this basic loading function need not be determined through solving the integral equation; instead it can easily be obtained by comparing the asymptotic value of the function at infinity with the two-dimensional load distribution.

Higher-order spanwise loading functions provide further modifications for the load distribution near the wing tip. We chose them to be

$$g_m(\bar{y}) = [\bar{y}^{1/2}\exp(-\bar{y})L_m(\bar{y})]\exp(i\bar{k}_y\bar{y}) \qquad \text{for} \qquad m \geqq 0 \tag{26}$$

where $L_m(\bar{y})$ is the mth Laguerre polynomial. As we know, the Laguerre polynomial expansion is just a different version of the power series expansion, except that it is more convenient for a semi-infinite interval.[14] The square-root factor and the exponential decay factor in Eq. (26) are multiplied to satisfy the side-edge condition and the asymptotic behavior.

Therefore, the assumed airload distribution can be taken as

$$\Delta\bar{C}_p(\bar{x},\bar{y}) = A_p\exp(i\bar{k}_y\bar{y})\bar{y}^{1/2}\exp(-\bar{y}) \times \left\{\left[(2\pi)^{1/2}I_0(\bar{y}) + \sum_{m=0} c_{0m}L_m(\bar{y})\right]\cot(\theta_c/2)\exp(K'\cos\theta_c) + \sum_{n=1}\left[\sum_{m=0} c_{nm}L_m(\bar{y})\right]\sin(n\theta_c)\right\} \tag{27}$$

which satisfies all the edge conditions, as well as the asymptotic condition at far inboard. The difference of the load distribution on the wing and that on the two-dimensional wing is then obtained as

$$\begin{aligned}\Delta\bar{C}_{p_c}(\bar{x},\bar{y}) &= \Delta\bar{C}_p - \Delta\bar{C}_{p_\infty} \\ &= A_p\exp(i\bar{k}_y\bar{y})\bar{y}^{1/2}\exp(-\bar{y})\left\{\left[(2\pi)^{1/2} \times \left(I_0(\bar{y}) - \frac{\exp(\bar{y})}{(2\pi\bar{y})^{1/2}}\right) + \sum_{m=0} c_{0m}L_m(\bar{y})\right] \times \right. \\ &\quad \cot(\theta_c/2)\exp(K'\cos\theta_c) + \left.\sum_{n=1}\left[\sum_{m=0} c_{nm}L_m(\bar{y})\right]\sin(n\theta_c)\right\}\end{aligned} \tag{28}$$

In expression (28), we can see that this load difference is decreasing with the exponential factor for increasing $\bar{y}$ or y/l_y. Thus, the choice of l_y influences the decay rate of this load difference or the range of the tip-influenced region. As mentioned before, l_y is a function of both the wing chord and the gust wavelength; if we consider these two characteristic lengths to be of the same order of magnitude, it is believed that the three-dimensional region influenced by the wing tip will extend a few chord lengths from the tip. Therefore, a reasonable first try would be $l_y = b_o$ or $\mu = 1$. This choice is justified by the convergence of the numerical solutions. Later results show that this choice is satisfactory for the semi-infinite wing with square tip.

Solution of the Integral Equation

By substituting the assumed series-form load distribution into the integral equation, and letting the normal velocity condition be satisfied at every control point, the integral equation is reduced to a set of linear algebraic equations. These equations are in the form

$$-8\pi\left[\frac{(\Delta\bar{w}-\bar{w}_b)}{U}\right]_j = \sum_{n=0}^{N-1}\left[\sum_{m=0}^{M-1}({}_jA_{nm})c_{nm}\right]; \qquad j = 1,2,\ldots,J;\; J \geqq N\times M \tag{29}$$

with N and M as the numbers of the chordwise and spanwise loading functions, and J, the number of the control points. The unknowns of the equations, c_{nm}, are the weighting factors of the loading functions. The coefficient of each unknown, ${}_jA_{nm}$, is just

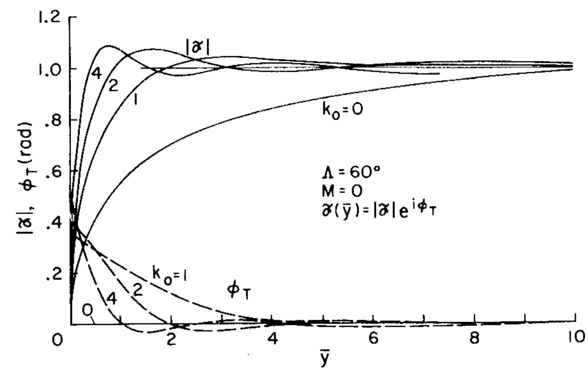

Fig. 2 Spanwise distribution of lift in terms of tip loading function.

the integration of the corresponding loading functions over the wing surface

$$\left.\begin{aligned} {}_jA_{nm} &= \oint_0^\infty \int_{-1}^1 f_n(\bar{x}_1)\bar{K}(\bar{x}_j-\bar{x}_1,\bar{y}_j-\bar{y}_1)\,d\bar{x}_1 \frac{g_m(\bar{y}_1)\,d\bar{y}_1}{(\bar{y}_j-\bar{y}_1)^2} \\ m &= 0,1,2,\ldots,M-1 \\ n &= 0,1,3,\ldots,N-1 \\ j &= 1,2,3,\ldots,J \end{aligned}\right\} \tag{30}$$

The term $\bar{w}_b$ in the left-hand side of Eq. (29) is the upwash induced by the load difference of the assumed basic loading function and the two-dimensional loading; namely

$$\left.\begin{aligned} &8\pi\left[-\frac{\bar{w}_b}{U}\right]_j = A_p \oint_0^\infty \left[\int_{-1}^1 \left(\frac{1-\bar{x}_1}{1+\bar{x}_1}\right)^{1/2} \times \right. \\ &\left.\exp(-K'\bar{x}_1)\bar{K}(\bar{x}_j-\bar{x}_1,\bar{y}_j-\bar{y}_1)\,d\bar{x}_1\right] \times \\ &[(2\pi\bar{y}_1)^{1/2}\exp(-\bar{y}_1)I_0(\bar{y}_1)-1]\exp(i\bar{k}_y\bar{y}_1)\frac{d\bar{y}_1}{(\bar{y}_j-\bar{y}_1)^2} \\ &j = 1,2,3,\ldots,J \end{aligned}\right\} \tag{31}$$

Since the coefficient of the basic loading function has already been determined by comparing it with the two-dimensional airload distribution, $\bar{w}_b$ is a known value and thus moved to the left-hand side of the equation.

The evaluation of the multi-integrals of ${}_jA_{nm}$ and $(\bar{w}_b)_j$ proceeds as follows: for the chordwise integration we use the Jacobi-Gauss quadrature formulas, with the weighting functions being $[(1-\bar{x})/(1+\bar{x})]^{1/2}$, $(1-\bar{x}^2)^{1/2}$, and so forth.[14,15] The number of the integration points is chosen to be either 10 or 20, according to the criterion discussed earlier in the direct problem. The spanwise integration can be performed by using the Legendre-Gauss and the Laguerre-Gauss quadrature formulas. Over the neighborhood of the singularity, the integral is evaluated in accordance with the Cauchy principal-value integral and the Mangler's principal-value integral.[13] Details of the procedure are given in Ref. 10.

Since $\Delta\bar{C}_{p_c}$ decays rapidly as $\bar{y}$ increases, the difficulties of slow decaying and fast oscillation which we encountered in the direct problem do not occur here. Therefore, the expansion of the kernel function is not necessary in the numerical integration.

With all coefficients of the simultaneous algebraic equation determined, the weighting factors for the loading functions can be found. If the number of the control points is chosen to be equal to that of the loading functions, these weighting factors can be determined by solving simultaneous algebraic equations. On the other hand, if the number of the control points is chosen to be more than that of the loading functions, results are obtained by using the least-squares method.[11]

Numerical Solutions and Discussions

In solving the integral equation, we chose 15 loading functions, which include 3 chordwise modes and 5 spanwise modes. We also chose 15 control points to satisfy the normal velocity condition. The positions of the control points are chosen in such a way that they are located at, or close to, the peaks of the loading functions. For the functions we use here, we chose the control points at $\bar{x} = -0.5, 0, 0.5$ and $\bar{y} = 0.2, 0.8, 1.5, 3, 5$.

Typical results for the spanwise lift distributions on the semi-infinite wing due to interacting with oblique sinusoidal gusts of various wavelengths are shown in Fig. 2 in terms of the tip-loading function $\mathscr{F}$, which is defined as

$$\mathscr{F}(\bar{y}; k_o, \Lambda, M) = C_L(\bar{y}; k_o, \Lambda, M)/C_{L_\infty}(k_o, \Lambda, M) \tag{32}$$

where C_L is the section lift coefficient of the wing and C_{L_∞} is that for a two-dimensional wing under the same gust conditions. From these results, we can see that the three-dimensional region influenced by the wing tip is reduced with increasing gust wave number. This effect can be interpreted by considering the contribution of the vorticity distributions in the wake. The gust-induced airload on a semi-infinite-span thin wing is influenced by three kinds of vorticity distributions in the wake: the first is the trailing vorticity due to the wing finiteness, the second is the trailing vorticity due to the air load oscillation along the span, and the third is the shed vorticity due to the unsteadiness. For increasing gust wave number, the second kind of trailing vorticity and the shed vorticity become more and more dominant and the weight of the first kind of trailing vorticity, which characterizes the three-dimensional effect of the wing tip, is reduced. Therefore, the tip-influenced region is also reduced. For large interaction angles, the trailing vorticity due to the spanwise airload oscillation makes the major contribution to this tip relief effect while for small interaction angles, the shed vorticity dominates. Therefore, the three-dimensional effect near the wing tip is important for low wave-number gusts. For wave numbers large enough, the three-dimensional tip effect can be neglected and the two-dimensional solution can be applied throughout the span.

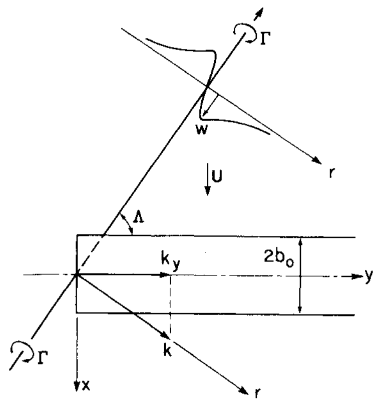

Fig. 3 Wing-vortex interaction at wing tip.

The interaction of the semi-infinite wing with a line vortex is studied as an example of practical application (Fig. 3). The vortex-induced upwash is first separated into a spectrum of sinusoidal components through Fourier analysis. Then, the induced airload can be obtained by summing up those due to all the components. The spanwise lift distributions, as well as the intersections of the vortex with the midchord at various times, is shown in Fig. 4. The two-dimensional solution for an infinite-span wing is also shown in the figure. By comparing these two cases, the three-dimensional effect of the wing tip should be very clear.

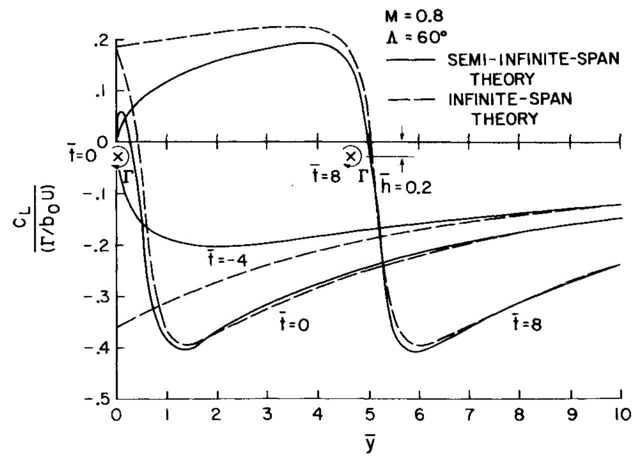

Fig. 4 Spanwise lift distribution on a semi-infinite wing—induced by an oblique line vortex.

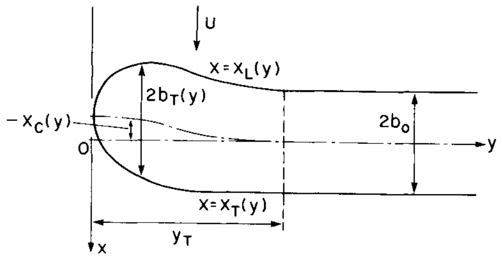

Fig. 5 Semi-infinite wing with arbitrary tip planform.

The lifting-surface theory developed here not only can be applied to a semi-infinite wing with square tip, but also can be extended for any arbitrary tip planform as well. For a semi-infinite wing shown in Fig. 5, the leading edge and the trailing edge can be described by the relations

$$\bar{x}_L = \begin{cases} \bar{x}_L(\bar{y}) & 0 \leqq \bar{y} < \bar{y}_T \\ -1 & \bar{y} \geqq \bar{y}_T \end{cases} \tag{33}$$

$$\bar{x}_T = \begin{cases} \bar{x}_T(\bar{y}) & 0 \leqq \bar{y} < \bar{y}_T \\ 1 & \bar{y} \geqq \bar{y}_T \end{cases} \tag{34}$$

where $\bar{y}_T$ shows the range of the arbitrary tip planform. The governing equation and the boundary conditions are the same as for the square tip discussed before, except that the locations at which the boundary conditions are applied have been changed. After converting the boundary value problem into an integral equation as before, it takes the form

$$\int_{-\infty}^{0}\int_{-1}^{1} \Delta\bar{C}_{p_\infty} K\, d\bar{x}_1\, d\bar{y}_1 + \int_{0}^{\bar{y}_T}\int_{-1}^{1} \Delta\bar{C}_{p_\infty} K\, d\bar{x}_1\, d\bar{y}_1 = \int_{0}^{\bar{y}_T}\int_{\bar{x}_L}^{\bar{x}_T} \Delta\bar{C}_p K\, d\bar{x}_1\, d\bar{x}_1 + \int_{\bar{y}_T}^{\infty}\int_{-1}^{1} \Delta\bar{C}_{p_c} K\, d\bar{x}_1\, d\bar{y}_1 \quad \text{for} \quad 0 < \bar{y} < \bar{y}_T \tag{35a}$$

or

$$\int_{-\infty}^{\bar{y}_T}\int_{-1}^{1} \Delta\bar{C}_{p_\infty} K\, d\bar{x}_1\, d\bar{y}_1 = \int_{0}^{\bar{y}_T}\int_{\bar{x}_L}^{\bar{x}_T} \Delta\bar{C}_p K\, d\bar{x}_1\, d\bar{y}_1 + \int_{\bar{y}_T}^{\infty}\int_{-1}^{1} \Delta\bar{C}_{p_c} K\, d\bar{x}_1\, d\bar{y}_1 \quad \text{for} \quad \bar{y} > \bar{y}_T \tag{35b}$$

The procedures to solve the integral equation are the same as what has been discussed earlier. However, there are two essential differences which require caution. First, because the locations of the leading and trailing edges were changed, the angular chordwise variable θ_c should be redefined as

$$\theta_c = \cos^{-1}\left\{-\frac{[x - x_c(y)]}{b_T(y)}\right\} \tag{36}$$

with

$$x_c(y) = \tfrac{1}{2}[x_T(y) + x_L(y)]$$

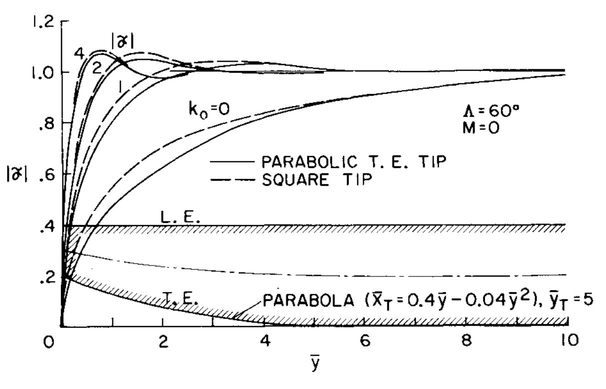

Fig. 6 Spanwise distribution of lift in terms of tip loading function.

and

$$b_T(y) = \tfrac{1}{2}[x_T(y) - x_L(y)]$$

as suggested in standard lifting-surface theories.[10] Second, the spanwise characteristic length l_y is now not only a function of the wing chord and the gust wavelength, but also a function of a third characteristic length y_T, the range of the arbitrary tip shape. This should be obvious because l_y represents a scale length for the range of the three-dimensional region near the tip, while the airload distribution on the wing won't converge to the two-dimensional loading until somewhere beyond y_T. If y_T is not too large and is of the same order of magnitude as b_o, we may still choose $l_y = b_o$ or $\mu = 1$. Figure 6 shows the gust-induced lift distributions for a wing tip with parabolic sweep-forward trailing edge; the square tip results are also plotted for comparison.

Conclusion

An unsteady lifting-surface theory has been developed for the calculation of the gust-induced airloads near the tip of a semi-infinite wing. It is found that the three-dimensional effect of the wing tip is reduced for increasing gust wave number. This reduction is attributable to the dominance of the shed and trailing vorticity in the wake due to time and spatial oscillation of the airload distribution over the trailing vorticity due to the wing finiteness. Therefore, for any gust-induced airload on the wing, the three-dimensional tip effect influences basically the low wavenumber components. For high wavenumber components, the wing tip has little effect and can just be neglected.

Numerical results in Fig. 6 indicate that the gust-induced dynamic airloads near the wing tip can be reduced by modifying the tip planform. This suggests a possible way to design the helicopter blade tip to relieve vibratory loads due to interaction of blades with trailing vortices. This will also reduce the noise due to blade-vortex interaction.

Appendix A

The expression of the modified form of the kernel function has been derived in Ref. 12 as

$$\bar{K}(\bar{x}_o, \bar{y}_o) = -\exp(is\bar{x}_o)\Bigg(is\mu|\bar{y}_o| + s\mu|\bar{y}_o| K_1(s\mu|\bar{y}_o|) - \frac{i\pi}{2} s\mu|\bar{y}_o| [I_1(s\mu|\bar{y}_o|) - L_1(s\mu|\bar{y}_o|)] + \frac{\bar{x}_o \exp\{-(is/\beta^2)[\bar{x}_o - M(\bar{x}_o^2 + \beta^2\mu^2\bar{y}_o^2)^{1/2}]\}}{(\bar{x}_o^2 + \beta^2\mu^2\bar{y}_o^2)^{1/2}} + is\mu|\bar{y}_o| \int_0^{\frac{1}{\beta^2\mu|\bar{y}_o|}[\bar{x}_o - M(\bar{x}_o^2 + \beta^2\mu^2\bar{y}_o^2)^{1/2}]} \times \frac{\tau}{(1+\tau^2)^{1/2}} \exp(-is\mu|\bar{y}_o|\tau)\, d\tau \Bigg). \tag{A1}$$

An alternate form is obtained in Ref. 10 as

$$\bar{K}(\bar{x}_o, \bar{y}_o) = -\exp(is\bar{x}_o)\Bigg[\!\!\Bigg[is\mu|\bar{y}_o| + s\mu|\bar{y}_o| K_1(s\mu|\bar{y}_o|) - is^2\mu^2\bar{y}_o^2 \Bigg\{ \frac{1.0085}{1.3410 + 1.0050 s^2\mu^2\bar{y}_o^2} + \left[\frac{\pi}{4} - 0.8675 s\mu|\bar{y}_o| \left(\frac{0.4648 + 0.9159 s\mu|\bar{y}_o|}{1.3410 + s^2\mu^2\bar{y}_o^2}\right)\right] \times \exp(-s\mu|\bar{y}_o|)\Bigg\} + \frac{\bar{x}_o \exp\{(-is/\beta^2)[\bar{x}_o - M(\bar{x}_o^2 + \beta^2\mu^2\bar{y}_o^2)^{1/2}]\}}{(\bar{x}_o^2 + \beta^2\mu^2\bar{y}_o^2)^{1/2}} + is\mu|\bar{y}_o| \sum_{m=1}^{5} \frac{q_m}{r_m - \varepsilon(is\mu|\bar{y}_o|)} \times [\![\exp\{[r_m - \varepsilon(is\mu|\bar{y}_o|)]\alpha_l\} - 1]\!]\Bigg]\!\!\Bigg] \tag{A2}$$

with

$$\alpha_l = (\varepsilon)[\bar{x}_o - M(\bar{x}_o^2 + \beta^2\mu^2|\bar{y}_o|^2)^{1/2}]/(\beta^2 s\mu|\bar{y}_o|)$$

$$q_m(m = 1, 5) = 1, \ -0.101, \ -0.899, 0.04740467i, \ -0.04740467i$$

$$r_m(m = 1, 5) = 0, \ -0.329, \ -1.4067, \ -2.90+\pi i, \ -2.90-\pi i$$

and

$$\varepsilon = \begin{cases} +1 & \text{for} \quad \bar{x}_o > M\mu|\bar{y}_o| \\ -1 & \text{for} \quad \bar{x}_o < M\mu|\bar{y}_o| \end{cases}$$

Appendix B

From Ref. 8, the airload distribution on a two-dimensional wing due to interaction with an oblique sinusoidal gust in a subsonic flow, under the conditions that the product of the flow Mach number and the reduced frequency is small and that the gust wavelength along the span is of the same order as the chord, has been found to be

$$\Delta C_{p_\infty}(k_o, \Lambda, M) =$$

$$A_p\left(\frac{1-\bar{x}}{1+\bar{x}}\right)^{1/2} \exp(-K'\bar{x})\exp[i(k_y y - \omega t)] \qquad \text{(B1)}$$

where

$$A_p \equiv \frac{4\hat{\alpha}}{\beta}\frac{T_{o_\infty}(K_o, \Lambda^T)}{I_0(K_2)+I_1(K_2)}$$

$$K_o = (k_o/\beta)[(\cos\Lambda/\beta)^2 + \sin^2\Lambda]^{1/2}$$

$$K_2 = K_o \sin\Lambda^T$$

$$K' = K_2 + i(M^2/\beta^2)s$$

$$\Lambda^T = \tan^{-1}(\beta\tan\Lambda)$$

The term T_{o_∞} is the aerodynamic transfer function for the incompressible flow, defined as

$$T_{o_\infty} \equiv \frac{C_L}{2\pi\hat{\alpha}\exp[i(k_y y - \omega t)]}$$

Its expression can be found in Ref. 4 as

$$T_{o_\infty}(k_o, \Lambda) = \frac{\exp\left\{-ik_o\left[\cos\Lambda - \dfrac{\pi(\pi/2-\Lambda)(1+\frac{1}{2}\sin\Lambda)}{1+2\pi k_o(1+\frac{1}{2}\sin\Lambda)}\right]\right\}}{[1+\pi k_o(1+\cos^2\Lambda + \pi k_o\sin\Lambda)]^{1/2}} \qquad \text{(B2)}$$

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